

# Private Approximate Heavy Hitters

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## Abstract

We consider the problem of private computation of approximate Heavy Hitters. Alice and Bob each hold a vector and, in the vector sum, they want to find the  $B$  largest values along with their indices. While the exact problem requires linear communication, protocols in the literature solve this problem approximately using polynomial computation time, polylogarithmic communication, and constantly many rounds. We show how to solve the problem *privately* with comparable cost, in the sense that nothing is learned by Alice and Bob beyond what is implied by their input, the ideal top- $B$  output, and goodness of approximation (equivalently, the Euclidean norm of the vector sum). We give lower bounds showing that the Euclidean norm must leak by any efficient algorithm.

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# 1 Introduction

Secure and private multiparty computation has been studied for several decades, starting with [20, 5]. Any protocol for computing a function of several inputs can be converted, gate-by-gate, to a *private* protocol, in which no party learns anything from the protocol messages other than what can be deduced from the function’s input/output relation. The computational overhead is at most polynomial in the size of the inputs.

In recent years, however, input sizes in many problems have grown to the point where “polynomial computational overhead” is too coarse a measure; both computation and communication should be minimized. For example, absent privacy concerns, applications may require that a protocol uses at most polylogarithmic communication. General-purpose secure multiparty computation may blow up communication exponentially, so additional techniques are needed. In one theoretical approach, individual protocols are designed for functions of interest such as database lookup (the *private information retrieval* problem [8, 17, 6]) and building decision trees [18]. Another approach, the breakthrough [19], converts any protocol into a private one with little communication blowup, but imposes a computational blowup that may be exponential.

The approach we follow, which was introduced in [10], is to substitute an approximate function for the desired function. Many functions of interest have good approximations that can be computed efficiently both in terms of computation and communication. A caveat is that the traditional definition of privacy is no longer appropriate. Instead, a protocol  $\pi$  computing an approximation  $\tilde{f}$  to a function  $f$  is a private approximation protocol [10] for  $f$  if

- $\pi$  is a private protocol for  $\tilde{f}$  in the traditional sense that the messages of  $\pi$  leak nothing beyond what is implied by inputs and  $\tilde{f}$ , *and*,
- the output  $\tilde{f}$  leaks nothing beyond what is implied by inputs and  $f$ .

Several examples were given in [10]. Another important example, crucial to this article and the first non-trivial example to achieve polylogarithmic communication and polynomial computation, was given in [14]. There, Alice and Bob have vectors  $a$  and  $b$  of length  $N$ , taking integer values in the range  $[-M, M]$ . Their goal is to approximate the Euclidean norm of the sum,  $\|a + b\|_2$ . The authors show how to compute an estimate  $\|a + b\|_\sim$  such that, if  $k$  is a security parameter,

- $\frac{1}{1+\epsilon}\|a + b\|_2 \leq \|a + b\|_\sim \leq \|a + b\|_2$ .
- The protocol requires  $\text{poly}(k \log(M)N/\epsilon)$  local computation,  $\text{poly}(k \log(M) \log(N)/\epsilon)$  communication, and  $O(1)$  rounds.
- No party learns more from the protocol messages than can be deduced from the approximate output  $\|a + b\|_\sim$  and the relevant party’s input, and no party learns more from the output  $\|a + b\|_\sim$  than can be deduced from the exact output  $\|a + b\|_2$ .

We will make use of this result.

## 1.1 Our Results

Each of two parties has a vector,  $a$  and  $b$ , and they want a summary for the vector sum  $c = a + b$ . First, we consider the Euclidean approximate heavy hitters problem, in which there is a parameter,  $B$ , and the players ideally want  $c_{\text{opt}}$ , the  $B$  largest terms in  $c$ , *i.e.*, the  $B$  biggest values together

with the corresponding indices. Unfortunately, finding  $c_{\text{opt}}$  exactly requires linear communication. Instead, the players use polylogarithmic communication (and polynomial work and  $O(1)$  rounds) to output a vector  $\tilde{c}$  with  $\|\tilde{c} - c\|_2 \leq (1 + \epsilon)\|c_{\text{opt}} - c\|_2$ . In our protocol, the players learn nothing more than what can be deduced from  $c_{\text{opt}}$  and  $\|c\|_2$ . (We discuss below the significance of leaking  $\|c\|_2$ .) We can immediately use this result as black box for approximate sparse representations over any orthonormal basis such as wavelet or Fourier, with similar costs. We can also use the result as a black box for taxicab approximate heavy hitters, *i.e.*, finding  $\tilde{c}$  with  $\|\tilde{c} - c\|_1 \leq (1 + \epsilon)\|c_{\text{opt}} - c\|_1$ , leaking  $c_{\text{opt}}$  and  $\|c\|_2$ .

In the basic result, we give an at-most- $B$ -term representation that is nearly as good (in the Euclidean sense) as the best  $B$ -term representation and leaks no more than the best  $B$ -term representation *and the Euclidean norm*. Leaking the Euclidean norm represents a weaker result than not leaking the Euclidean norm, but (i) leaking  $\|c\|_2$  is necessary in some circumstances and (ii) computing or approximating  $\|c\|_2$  is desirable in some circumstances. First, we give a straightforward lower bound showing that, for some (reasonable) values of parameters  $M, N, \dots$ , computing  $\tilde{c}$  leaking only  $c_{\text{opt}}$  requires  $\Omega(N)$  communication. In fact, for some (artificial) classes of inputs,  $\Omega(N)$  communication is needed unless  $\|c\|_2$  itself is not only potentially leaked, but actually computed exactly. On the other hand, one can regard the Euclidean norm as semantically interesting, so that we can regard the top  $B$  terms *together with the Euclidean norm* as a compound, extended summary. In particular, since  $\tilde{c}$  is computed, leaking  $\|c\|_2$  is equivalent to leaking  $\|c\|_2^2 - \|\tilde{c}\|_2^2 = \|\tilde{c} - c\|_2^2$ , *i.e.*, the error in our representation, which is a useful and common thing to want to compute. Our protocol indeed can be modified to output an approximation  $\|\tilde{c} - c\|_\sim$  with  $\|\tilde{c} - c\|_2 \leq \|\tilde{c} - c\|_\sim \leq (1 + \epsilon)\|\tilde{c} - c\|_2$ , so we can regard the protocol as solving two cascaded approximation problems: find a near-best representation  $\tilde{c}$ , then find an approximation  $\|\tilde{c} - c\|_\sim$  to  $\|\tilde{c} - c\|_2$ . It is natural to expect a protocol for  $\tilde{c}$  to leak  $c_{\text{opt}}$  and a protocol for  $\|\tilde{c} - c\|_\sim$  to leak  $\|\tilde{c} - c\|_2$ ; while lower bounds prevent that, we can compute  $\tilde{c}$  and  $\|\tilde{c} - c\|_\sim$  *simultaneously* and guarantee that, *overall*, we leak only  $c_{\text{opt}}$  and  $\|\tilde{c} - c\|_2$ .

We give a result for taxicab heavy hitters that produces an at-most- $B$  term representation that is nearly as good (in the taxicab sense) as the the best  $B$ -term representation and leaks no more than the best  $B$ -term representation and the *Euclidean* norm. Thus we have shown that the private Euclidean norm approximation can be used for non-Euclidean problems. Finally, we also give a result for other orthonormal bases that involves little additional algorithmic or privacy work, but demonstrates that the basic result can be applied in a variety of interesting applications. It says that we provide an at-most- $B$  term Fourier representation that is almost as good (in the Euclidean sense) as the best  $B$ -term Fourier representation and leaks no more than the best  $B$ -term representation and the Euclidean norm. The Fourier basis may be substituted by any orthonormal basis, such as Hadamard or Wavelet.

## 1.2 Related Work

Other work in private communication-efficient protocols for specific functions includes the Private Information Retrieval problem [8, 17, 6], building decision trees [18], set intersection and matching [11], and  $k$ 'th-ranked element [1].

The breakthrough [19] gives a general technique for converting any protocol into a private protocol with little communication overhead. It is not the end of the story, however, because the computation may increase exponentially.

Work in private approximations include [10] that introduced the notion as a conference paper

in 2001 and gave several protocols. Some negative results were given in [13] for approximations to NP-Hard functions; more on NP-hard search problems appears in [4]. Recently, [14] gives a private approximation to the Euclidean norm that is central to our paper.

Statistical work such as [7] also addresses approximate summaries over large databases, but differs from our work in many parameters, such as the number of players and the allowable communication.

There are many papers that address the Heavy Hitters problem and sketching in general, in a variety of contexts. Many of the needed ideas can be seen in [15] and other important papers include [3, 2, 12, 9].

### 1.3 Organization

This paper is organized as follows. In Section 2, we give preliminaries. In Section 3, we present our main result. In Section 4, we present lower bounds.

## 2 Preliminaries

### 2.1 Parameters and Notation

Fix parameters  $N, M, B, k, \epsilon$ . We will consider two players, Alice and Bob, who will have inputs,  $a$  and  $b$  respectively, that are vectors of length  $N$  taking integer values in the range  $-M$  to  $+M$ . Throughout, we will be interested in summaries of size  $B$  for the vector  $c = a + b$ . For example, in the main result, we are interested ideally in the largest  $B$  terms of  $c$ . A vector  $c$  is written  $c = (c_0, c_1, c_2, \dots, c_{N-1}) = \sum c_j \delta_j$ , where  $j$  is an *index*,  $c_j$  is a *value*,  $\delta_j$  is the vector that is 1 at index  $j$  and 0 elsewhere, and  $c_j \delta_j$ , which can be implemented compactly and equivalently written as the pair  $(j, c_j)$ , is a *term*, in which  $c_j$  is the *coefficient*.

We compare terms by the *magnitudes* of their coefficients, breaking ties by the indices. That is, we will say that  $(j, c_j) < (k, c_k)$  if  $|c_j| < |c_k|$  or both  $|c_j| = |c_k|$  and  $j < k$ . Thus all terms are strictly comparable. A heavy hitter summary is an expression of the form  $\sum_{i \in \Lambda} \eta_i \delta_i$ . If  $|\Lambda|$  must be at most  $B$ , then the best heavy hitter summary  $c_{\text{opt}}$  for a vector  $c$  occurs where  $\{(i, \eta_i) : i \in \Lambda\}$  consists of the  $B$  largest terms.

The Euclidean norm of  $a$  is  $\|a\|_2 = \sqrt{\sum_i a_i^2}$  and the taxicab norm is  $\|a\|_1 = \sum_i |a_i|$ . The *support*  $\text{supp}(a)$  of a vector  $a$  is the set of indices where  $a$  is non-zero,  $\{i : a_i \neq 0\}$ .

The parameter  $\epsilon$  is a distortion parameter. We will guarantee summaries whose error is at most the factor  $(1 + \epsilon)$  times the error of the best possible summary.

The parameter  $k$  is a security and failure probability parameter. Algorithms will be expected to succeed except with probability  $2^{-k}$  and  $2^{-k}$  will serve as an upper bound for the allowable statistical distance between indistinguishable distributions.

We will be interested in protocols that use communication  $\text{poly}(B, \log(N), k, \log(M), 1/\epsilon)$ , local computation  $\text{poly}(B, N, k, \log(M), 1/\epsilon)$ , and number of rounds that is constant.

### 2.2 Approximate Data Summaries

In the heavy hitters problem, we are given parameters  $B$  and  $N$  and the goal is to find the  $B$  largest terms in a vector  $c$  of length  $N$ . We will be interested in two approximate versions, parametrized also by  $\epsilon$ . In the approximate heavy hitters problem, we want a summary  $\tilde{c} = \sum_{i \in \Lambda} \eta_i \delta_i$  such that

$\|\tilde{c} - c\| \leq (1 + \epsilon)\|c_{\text{opt}} - c\|$ , where the norms are, respectively, 2-norms (in the Euclidean approximate heavy hitters problem) and 1-norms (in the taxicab approximate heavy hitters problem).

In order to describe previous algorithms that are relevant to us, we first need some definitions. Fix a vector  $c = (c_0, c_1, c_2, \dots, c_{N-1}) = \sum_{0 \leq i < N} c_i \delta_i$ , whose terms are  $t_0 = (0, c_0), t_1 = (1, c_1), \dots, t_{N-1} = (N-1, c_{N-1})$ . Suppose the sequence  $i'_0, i'_1, \dots$  is a decreasing rearrangement of  $c$ , i.e.,  $t_{i'_0} > t_{i'_1} > \dots > t_{i'_{N-1}}$ .

**Definition 2.1** (*Significant index.*) Let  $I \subseteq [0, N)$  be a set of indices containing  $i$ . Then  $i$  is a  $(I, \theta)$ -significant index for  $c$  if and only if  $c_i^2 \geq \theta \sum_{j \in I} |c_j|^2$ .

That is, an index is significant if the corresponding value is large compared with all the values. In some of the algorithms below, we will find the largest term (if it is sufficiently large), subtract it off, then recurse on the residual signal. This motivates the following definitions.

**Definition 2.2** (*Significant index set.*) Let  $I \subseteq [0, N)$  be a set of  $m$  indices containing  $i$ . Then  $I$  is a  $\theta$ -significant index set for  $c$  if and only if  $\forall j = 1 \dots m$ ,  $t_{i'_j}$  is a  $([0, N) \setminus \{i'_1, \dots, i'_{j-1}\}, \theta)$ -significant index.

That is, in a significant index set for  $c$ , the largest term has a significant index; after removing the largest term, the new largest term has a significant index, etc. Note that there can be more than one  $\theta$ -significant index set for a given vector.

**Definition 2.3** (*Qualified index set.*) Fix parameters  $\ell$  and  $\theta$ . The set  $Q = \{i'_0, i'_1, \dots, i'_{m-1}\}$  is a  $(\ell, \theta)$ -qualified index set for  $c$  if and only if

- $m \leq \ell$ ,
- $\{i'_0, i'_1, \dots, i'_{m-1}\}$  is a  $\theta$ -significant index set, and
- $\{i'_0, i'_1, \dots, i'_{m-1}, i'_m\}$  is NOT a  $\theta$ -significant index set.

That is, a qualified index set consists of the largest possible length  $m$  for a prefix of  $i'_0, i'_1, \dots, i'_{m-1}$  such that, for each  $j < m$ , we have  $c_{i'_j}^2 \geq \theta(c_{i'_j}^2 + c_{i'_{j+1}}^2 + c_{i'_{j+2}}^2 + \dots + c_{i'_{N-1}}^2)$ . In particular, if the terms happen to be in decreasing order to begin with, i.e., if  $|c_0| > |c_1| > \dots$ , then a qualified index set is  $\{0, 1, 2, \dots, m-1\}$  for the largest  $m$  such that, for each  $j < m$ , we have  $c_j^2 \geq \theta(c_j^2 + c_{j+1}^2 + c_{j+2}^2 + \dots + c_{N-1}^2)$ .

Note that for each  $\ell, \theta$ , and vector  $c$ , there is only one  $(\ell, \theta)$ -qualified index set for  $c$ . We use  $Q_{c, \ell, \theta}$  to denote it. We sometimes write  $Q_{\ell, \theta}$  when  $c$  is understood.

The following are straightforward.

**Proposition 2.4** For any  $\theta_1 < \theta_2$ ,  $Q_{\ell, \theta_2}$  set is a subset of  $Q_{\ell, \theta_1}$ .

**Proposition 2.5** Fix parameters  $N, M, B, k, \epsilon$  and vector  $c$  as above. If  $\tilde{c} = \sum_{i \in Q_{c, B, \frac{\epsilon}{B(1+\epsilon)}}} c_i \delta_i$ , then  $\|\tilde{c} - c\|_2^2 \leq (1 + \epsilon)\|c_{\text{opt}} - c\|_2^2$ .

**Proof:** Assume without loss of generality that  $|c_0| > |c_1| > \dots$  and let  $q = |Q_{c,B,\frac{\epsilon}{B(1+\epsilon)}}|$ . If  $q = B$ , then  $\tilde{c} = c_{\text{opt}}$  and we are done. Otherwise we have

$$\begin{aligned} \|\tilde{c} - c\|_2^2 &= \sum_{q \leq i < B} |c_i|^2 + \|c_{\text{opt}} - c\|_2^2 \\ &\leq B|c_q|^2 + \|c_{\text{opt}} - c\|_2^2 \\ &\leq \frac{\epsilon}{1+\epsilon} \|\tilde{c} - c\|_2^2 + \|c_{\text{opt}} - c\|_2^2, \end{aligned}$$

whence

$$\left(1 - \frac{\epsilon}{1+\epsilon}\right) \|\tilde{c} - c\|_2^2 \leq \|c_{\text{opt}} - c\|_2^2.$$

The result follows. ■

The algorithms below will work from a linear *sketch* of a vector.

**Definition 2.6** (*Sketch of a vector.*) Given a vector  $c$ , a linear sketch of  $c$  is  $Rc$ , where  $R$  is a random matrix generated from a prescribed distribution, called the measurement matrix.

In our case, as is typical, the matrix  $R$  will be a pseudorandom matrix, that can be generated from a short pseudorandom seed. We will use sketching for the NORM\_ESTIMATION protocol, in which the generator needs to be secure against small space, and a different measurement matrix in the non-private Euclidean Heavy Hitters protocol, where, *e.g.*, pairwise independence suffices for the pseudorandom number generator.

An algorithm in connection with the Euclidean approximate heavy hitter problem satisfying the following is known:

**Theorem 2.7** Fix parameters  $N, M, B, k, \epsilon$  as above. Fix  $\theta \geq \text{poly}(\log(N), \log(M), B, k, 1/\epsilon)^{-1}$ . There is a distribution on sketch matrices  $R$  and a corresponding algorithm that, from  $R$  and sketch  $Rc$  of a vector  $c$ , outputs a superset of  $Q_{c,B,\theta}$ , in time  $\text{poly}(\log(N), \log(M), B, k, 1/\epsilon)$ .

In particular, the number of rows in  $R$  and the size of the output is bounded by the expression  $\text{poly}(\log(N), \log(M), B, k, 1/\epsilon)$  in accordance with the time bound on the algorithm.

Note that the algorithm returns a superset of  $Q_{c,B,\theta}$  but that even  $Q_{c,B,\theta}$  itself suffices for a good approximation.

**Proof:** [sketch] One such algorithm is as follows. As in [12], one can estimate  $c_i$  by  $\tilde{c}_i = \delta_i^T R^T Rc \pm (\epsilon/B)\|c\|_2$  except with small probability, where  $R$  is a  $\pm 1$ -valued matrix with  $\text{poly}(\log(N), B, 1/\epsilon)$  independent rows, each of which is a pairwise independent family. By repeating  $O(k)$  times and taking a median, one can drive down the failure probability to  $2^{-k}$ . As in [12], one need not estimate all the terms; rather, in time  $\text{poly}(\log(N), \log(M), B, k, 1/\epsilon)$ , one can find a set  $I$  of indices that includes all terms with magnitude at least  $\theta\|c\|_2$  (and possibly other terms). By adjusting parameters, one can estimate such  $c_i$  well enough as  $\tilde{c}_i$  so that  $|\tilde{c}_i - c_i|^2 \leq (\epsilon/B)\|c\|_2^2$ . To get a superset of a qualified set, subtract off the approximation to  $c_i\delta_i$  and repeat as long as new  $c_i$  (or better approximations to old  $c_i$ ) are found that are large compared with the residual vector. At most  $O(\log(MN))$  repetitions are needed since, after  $O(\log(MN))$  repetitions, we have reduced  $\|c\|_2^2$  from its initial value of at most  $M^2N$  to its least possible positive value of 1. ■

## 2.3 Privacy

Secure multiparty computation allows two or more parties to evaluate a specified function of their inputs while hiding their inputs from each other. We work in the semi-honest model, which assumes that the adversary is passive and can't modify the behavior of corrupted parties. In particular, the computation is only concerned with the information learned by the adversary, and not with the effect misbehavior may have on the protocol's correctness.

We briefly review private two-player protocols in the semi-honest model. A two-party computation task is specified by a (possibly randomized) mapping  $g$  from a pair of inputs  $(a, b) \in \{0, 1\}^* \times \{0, 1\}^*$  to a pair of outputs  $(c, d) \in \{0, 1\}^* \times \{0, 1\}^*$ . Let  $\pi = (\pi_A, \pi_B)$  be a two-party protocol computing  $g$ . Consider the probability space induced by the execution of  $\pi$  on input  $\mathbf{x} = (a, b)$  (induced by the independent choices of random inputs  $r_A, r_B$ ). Let  $\text{view}_A^\pi(\mathbf{x})$  (resp.,  $\text{view}_B^\pi(\mathbf{x})$ ) denote the entire view of Alice (resp., Bob) in this execution, including her input, random input, and all messages she has received. Let  $\text{output}_A^\pi(\mathbf{x})$  (resp.,  $\text{output}_B^\pi(\mathbf{x})$ ) denote Alice's (resp., Bob's) output. Note that the above four random variables are defined over the same probability space. Two distributions (or ensembles)  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are said to be *computationally indistinguishable* with security parameter  $k$ ,  $\mathcal{D}_1 \stackrel{c}{\equiv} \mathcal{D}_2$ , if, whenever  $X_1 \sim \mathcal{D}_1$  and  $X_2 \sim \mathcal{D}_2$  and for any function  $C$  having a circuit of size at most  $2^k$ , we have then  $|\Pr(C(X_1) = 1) - \Pr(C(X_2) = 1)| \leq 2^{-k}$ .

**Definition 2.8** *Let  $X$  be the set of all valid inputs  $\mathbf{x} = (a, b)$ . A protocol  $\pi$  is a private protocol computing  $g$  if the following properties hold:*

**Correctness.** *The joint outputs of the protocol are distributed according to  $g(a, b)$ . Formally,*

$$\{(\text{output}_A^\pi(\mathbf{x}), \text{output}_B^\pi(\mathbf{x}))\}_{\mathbf{x} \in X} \equiv \{(g_A(\mathbf{x}), g_B(\mathbf{x}))\}_{\mathbf{x} \in X},$$

*where  $(g_A(\mathbf{x}), g_B(\mathbf{x}))$  is the joint distribution of the outputs of  $g(\mathbf{x})$ .*

**Privacy.** *There exist probabilistic polynomial-time algorithms  $\mathcal{S}_A, \mathcal{S}_B$ , called simulators, such that:*

$$\begin{aligned} \{(\mathcal{S}_A(a, g_A(\mathbf{x})), g_B(\mathbf{x}))\}_{\mathbf{x}=(a,b) \in X} &\stackrel{c}{\equiv} \{(\text{view}_A^\pi(\mathbf{x}), \text{output}_B^\pi(\mathbf{x}))\}_{\mathbf{x} \in X} \\ \{g_A(\mathbf{x}), \mathcal{S}_B(b, g_B(\mathbf{x}))\}_{\mathbf{x}=(a,b) \in X} &\stackrel{c}{\equiv} \{(\text{output}_A^\pi(\mathbf{x}), \text{view}_B^\pi(\mathbf{x}))\}_{\mathbf{x} \in X} \end{aligned}$$

There are efficient general techniques:

**Proposition 2.9** *(General-Purpose Secure Multiparty Computation (SMC) [20]) Two parties holding inputs  $x$  and  $y$  can privately compute any circuit  $C$  with communication and computation  $O(k(|C| + |x| + |y|))$ , where  $k$  is a security parameter, in  $O(1)$  rounds.*

Private approximation requires further discussion.

**Definition 2.10 (Private Approximation Protocol ([10]))** *A two-party protocol  $\pi$  is a private approximation protocol for a deterministic, common-output function  $g$  on inputs  $a$  and  $b$  if  $\pi$  computes a (possibly randomized) approximation  $\tilde{g}$  to  $g$  such that*

- $\tilde{g}$  is a good approximation to  $g$  (in the appropriate sense)
- $\pi$  is a private protocol for  $\tilde{g}$  in the traditional sense.

- (Functional Privacy.) There exists a probabilistic polynomial-time simulator  $\mathcal{S}$  such that:

$$\{\mathcal{S}(g(\mathbf{x}))\}_{\mathbf{x}=(a,b) \in X} \stackrel{c}{=} \tilde{g}(\mathbf{x}).$$

In our case,  $g(a, b)$  will formally be the pair  $(c_{\text{opt}}, \|c\|_2)$  and  $\tilde{g}(a, b)$  will be  $\tilde{c}$ . We will informally say that we “approximate  $c_{\text{opt}}$  leaking only  $c_{\text{opt}}$  and  $\|c\|_2$ ,” since there is a simulator that takes  $c_{\text{opt}}$  and  $\|c\|_2$  as input and simulates the approximate output  $\tilde{c}$  and the protocol messages. Equivalently, one could define  $g(a, b)$  to be the pair  $(c_{\text{opt}}, \|c_{\text{opt}} - c\|)$  and define  $\tilde{g}(a, b)$  to be the pair  $(\tilde{c}, \|\tilde{c} - c\|_{\sim})$ , where  $\|\cdot\|_{\sim}$  is an approximation to the Euclidean norm (see below).

In our case of a deterministic function to be output to both Alice and Bob, a (weakly) equivalent definition is as follows, known as the “liberal” definition in [10]:

**Definition 2.11** *A two-party protocol  $\pi$  is a private approximation protocol for a deterministic, common-output function  $g$  on inputs  $a$  and  $b$  in the liberal sense if  $\pi$  computes a (possibly randomized) approximation  $\hat{g}$  to  $g$  such that*

- $\hat{g}$  is a good approximation to  $g$  (in the appropriate sense)
- There exists a probabilistic polynomial-time simulators  $\mathcal{S}_A$  and  $\mathcal{S}_B$  such that:

$$\begin{aligned} \{\mathcal{S}_A(a, g(\mathbf{x}))\}_{\mathbf{x}=(a,b) \in X} &\stackrel{c}{=} \{\text{view}_A^\pi(\mathbf{x})\}_{\mathbf{x} \in X} \\ \{\mathcal{S}_B(b, g(\mathbf{x}))\}_{\mathbf{x}=(a,b) \in X} &\stackrel{c}{=} \{\text{view}_B^\pi(\mathbf{x})\}_{\mathbf{x} \in X} \end{aligned}$$

Roughly speaking, the equivalence is as follows. Suppose there are simulators in the standard definition. Then, putting  $\hat{g} = \tilde{g}$ , a simulator for the liberal definition can be constructed by simulating  $\hat{g}(a, b) = \tilde{g}(a, b)$  from  $g(a, b)$  using the hypothesized simulator for functional privacy, then simulating Alice’s view from  $\hat{g}(a, b)$  and  $a$  using the hypothesized simulator traditional simulator for the protocol that computes  $\tilde{g}$ . In the other direction, suppose there is a simulator in the liberal definition. Let  $\tau$  be a transcript of Alice’s view except for input  $a$ . (As it turns out, it is not necessary to include  $a$  in  $\tau$ . If  $a$  is much longer than  $\tau$ —as in our situation—we want to avoid including  $a$  in  $\tau$  in order to keep  $\tau$  short.) Define  $\tilde{g} = \hat{g}.\tau$  to be  $\hat{g}$  with  $\tau$  encoded into its low-order bits. We assume that this kind of encoding into approximations can be accomplished without significantly affecting the goodness of approximation; in fact, we will assume that the value represented does not change at all, even if the “approximate” value is zero—that is,  $\tau$  is auxiliary data rather than an actual part of the value of  $\tilde{g}$ . Note that a protocol for  $\hat{g}$  also serves as a protocol for  $\tilde{g}$ . It is trivial to simulate the messages of the protocol given  $a$  and  $\tilde{g}$ . Use the hypothesized simulator in the liberal definition to show functional privacy.

We will use the technique of encoding into the low-order bits in our main result, which, formally, will be proven in the standard definition. We remark that the NORM\_ESTIMATION protocol from [14] is presented in the liberal definition.

We will need the following standard definition.

**Definition 2.12 (Additive Secret Sharing)** *An intermediate value  $x$  of a joint computation is said to be secret shared between Alice and Bob if Alice holds  $r$  and Bob holds  $x - r$ , modulo some large prime, where  $r$  is a random number independent of all inputs and outputs.*



The Private Sample Sum problem is as follows.

**Definition 2.13 (Private Sample Sum)** *At the start, Alice holds a vector  $a$  of length  $N$  and Bob holds a vector  $b$ . Alice and Bob also hold a secret sharing of an index  $i$ . At the end, Alice and Bob hold a secret sharing of  $a_i + b_i$ .*

That is, neither the index  $i$  nor the value  $a_i + b_i$  becomes known to the parties. Efficient protocols for this can be found (or can be constructed immediately from related results) in [19, 10], under various assumptions about the existence of Private Information Retrieval, such as in [6].

**Proposition 2.14** *There is a protocol PRIVATE-SAMPLE-SUM for the Private Sample Sum problem that requires  $\text{poly}(N, k)$  computation,  $\text{poly}(\log(N), k)$  communication, and  $O(1)$  rounds.*

Our results also rely on the following protocol from [14], that privately approximates the Euclidean norm of the vector sum.

**Proposition 2.15 (Private  $l_2$  approximation)** [14] *Suppose Alice and Bob have integer-valued vectors  $a$  and  $b$  in  $[-M, M]^N$  and let  $c = a + b$ . Fix distortion  $\epsilon$  and security parameter  $k$ . There is a protocol NORM\_ESTIMATION that computes an approximation  $\|c\|_\sim$  to  $\|c\|_2$  such that*

- $\frac{1}{1+\epsilon} \|a + b\|_2 \leq \|a + b\|_\sim \leq \|a + b\|_2$ .
- *The protocol requires  $\text{poly}(k \log(M)N/\epsilon)$  local computation,  $\text{poly}(k \log(M) \log(N)/\epsilon)$  communication, and  $O(1)$  rounds.*
- *The protocol is a private approximation protocol for  $\|c\|$  in the sense of Definition 2.11.*

Furthermore, the protocol's only access to  $a$  and  $b$  is through the matrix-vector products  $Ra$  and  $Rb$ , where  $R$  is a pseudorandom matrix known to both players.

### 3 Private Euclidean Heavy Hitters

We consider the setting in which Alice has signal  $a$  of dimension  $N$ , and Bob has signal  $b$  of the same dimension. Let  $c = a + b$ . Both parties want to learn a representation  $\tilde{c} = \sum_{t \in T_{\text{out}}} t$  such that  $\|c - \tilde{c}\|_2^2 \leq (1 + \epsilon) \|c - c_{\text{opt}}\|_2^2$  and such that at most  $c_{\text{opt}}$  and  $\|c\|_2$  is revealed. A protocol is given in Figure 1.

#### 3.1 Analysis

First, to gain intuition, we consider some easy special cases of the protocol's operation. For our analysis, assume that the terms in  $c$  are already positive and in decreasing order,  $c_0 > c_1 > \dots > c_{N-1} > 0$ . We will be able to find the coefficient value of any desired term, so we focus on the set of indices. Let  $I_{\text{opt}} = \{0, 1, 2, \dots, B-1\}$  denote the set of indices for the optimal  $B$  terms. Thus  $Q_{c,B,\theta} \subseteq Q_{c,B,\frac{\theta}{1+\epsilon}} \subseteq I_{\text{opt}}$  and  $Q_{c,B,\frac{\theta}{1+\epsilon}} \subseteq I$ .

The ideal output is  $I_{\text{opt}}$ , though any superset of  $Q_{c,B,\theta}$  suffices to get an approximation with error at most  $(1 + \epsilon)$  times optimal. This includes the set  $I \supseteq Q_{c,B,\theta}$  which the algorithm has recovered. The set  $I_B$  of the largest  $B$  terms indexed by  $I$  contains  $Q_{c,B,\theta}$ , so  $I_B$  is a set of at most

PRIVATE\_EUCLIDEAN\_HEAVY\_HITTERS

- Known structural parameters:  $N, M, B, \epsilon, k$ , which determine  $\theta = \frac{\epsilon}{B(1+\epsilon)}$  and  $B'$
- Individual inputs: vectors  $a$  and  $b$ , of length  $N$ , with integer values in the range  $[-M, M]$ .
- Output: With probability at least  $1 - 2^{-k}$ , a set  $T_{\text{out}}$  of at most  $B$  terms, such that  $\left\|c - \sum_{t \in T_{\text{out}}} t\right\|_2^2 \leq (1 + \epsilon) \left\|c - \sum_{t \in T_{\text{opt}}} t\right\|_2^2$ .

- 
1. Exchange pseudorandom seeds (in the clear). Generate measurement matrices  $R_1$  and  $R_2$ . Alice locally constructs sketches  $R_1a$  and  $R_2a = (R_2^0a, R_2^1a, \dots, R_2^{B-1}a)$ , where the measurement matrix  $R_1$  is used for a non-private Euclidean Heavy Hitters and the measurement matrix  $R_2 = (R_2^0, R_2^1, \dots, R_2^{B-1})$  is used for  $B$  independent repetitions of NORM\_ESTIMATION. Bob similarly constructs  $R_1b$  and  $R_2b$ .
  2. Using general-purpose SMC, do
    - Use an existing (non-private) Euclidean Heavy Hitters protocol to get, from  $R_1a$  and  $R_1b$ , a secret-sharing of a superset  $I$  of  $Q_{c,B,\frac{\theta}{1+\epsilon}}$ , in which  $I$  has exactly  $B' \leq \text{poly}(\log(N), \log(M), B, k, 1/\epsilon)$  indices. (Pad with arbitrary indices, if necessary.)
  3. Use PRIVATE-SAMPLE-SUM to compute, from  $I, a$ , and  $b$ , secret-shared values for each index in  $I$ . Let  $T$  denote the corresponding set of secret-shared terms. (Both the index and value of each term in  $T$  is secret shared.) Enumerate  $I$  as  $I = \{i_0, i_1, \dots\}$  with  $t_{i_0} > t_{i_1} > \dots$ .
  4. Using SMC, do
    - for  $j = 0$  to  $B - 1$ 
      - (a) From  $R_2^j, R_2^ja, R_2^jb, t_0, t_1, \dots, t_{i_{j-1}}$ , sketch  $r_j = c - (t_{i_0} + t_{i_1} + \dots + t_{i_{j-1}})$  as  $R_2^jr_j = (R_2^ja + R_2^jb - R_2^j(t_{i_0} + t_{i_1} + \dots + t_{i_{j-1}}))$ .
      - (b) use NORM\_ESTIMATION to estimate  $\|r_j\|_2^2$  as  $\|r_j\|_\sim^2$ , satisfying  $\frac{1}{1+\epsilon}\|r_j\|_2^2 \leq \|r_j\|_\sim^2 \leq \|r_j\|_2^2$ .
      - (c) If  $|c_{i_j}|^2 < \theta\|r_j\|_\sim^2$ , break (out of for-loop)
      - (d) Output  $t_j$
  5. For technical reasons, encode the pseudorandom seeds for  $R_1$  and  $R_2$  into the low-order bits of the output or (as we assume here) provide  $R_1$  and  $R_2$  as auxiliary output.

Figure 1: Protocol for the Euclidean Heavy Hitters problem

$B$  terms with error at most  $(1 + \epsilon)$  times optimal. If  $|Q_{c,B,\theta}| = B$ , then  $I_B = Q_{c,B,\theta} = I_{\text{opt}}$ , and  $I_B$  is a private and correct output.

The difficulty arises when  $|Q_{c,B,\theta}| < B$ , in which case some of  $I_B$  may be arbitrary and should not be allowed to leak. So the algorithm needs to find a private subset  $I_{\text{out}}$  with  $Q_{c,B,\theta} \subseteq I_{\text{out}} \subseteq I_B$ . The challenge is subtle. Let  $s$  denote  $|Q_{c,B,\theta}|$ . If the algorithm knew  $s$ , the algorithm could easily output  $Q_{c,B,\theta}$ , which is the indices of the top  $s$  terms, a correct and private output. Unfortunately, determining  $Q_{c,B,\theta}$  or  $s = |Q_{c,B,\theta}|$  requires  $\Omega(N)$  communication (see Section 4), so we cannot hope to find  $Q_{c,B,\theta}$  exactly. Non-private norm estimation can be used to find a subset  $I_{\text{out}}$  with  $Q_{c,B,\theta} \subseteq I_{\text{out}} \subseteq Q_{c,B,\frac{\theta}{1+\epsilon}} \subseteq I_{\text{opt}}$ , which is correct, but not quite private. Given  $|I_{\text{out}}|$ , the contents of  $I_{\text{out}} \subseteq I_{\text{opt}}$  are indeed private, but the *size* of  $I_{\text{out}}$  is, generally, non-private. Fortunately, if we use a *private* protocol for norm estimation,  $|I_{\text{out}}|$  remains private. We now proceed to a formal analysis.

**Theorem 3.1** *Protocol PRIVATE\_EUCLIDEAN\_HEAVY\_HITTERS requires  $\text{poly}(N, \log(M), B, k, 1/\epsilon)$  local computation,  $\text{poly}(\log(N), \log(M), B, k, 1/\epsilon)$  communication, and  $O(1)$  rounds.*

**Proof:** By existing work, all costs of Steps 1 to 3 are as claimed. Now consider Step 4. Observe that the function being computed in Step 4 has inputs and outputs of size bounded by  $\text{poly}(\log(N), \log(M), B, k, 1/\epsilon)$  and takes time polynomial in the size of its inputs. In particular, the instances of NORM\_ESTIMATION do *not* start from scratch with a reference to  $a$  or  $b$ ; rather, they pick up from the precomputed short sketches  $R_2a$  and  $R_2b$ . It follows that this function can be wrapped with SMC, preserving the computation and communication up to polynomial blowup in the size of the input and keeping the round complexity to  $O(1)$ . ■

We now turn to correctness and privacy. Let  $I_{\text{out}}$  denote the set of indices corresponding to the set  $T_{\text{out}}$  of output terms.

**Theorem 3.2** *Protocol PRIVATE\_EUCLIDEAN\_HEAVY\_HITTERS is correct.*

**Proof:** The correctness of Steps 2 and 3 follows from previous work. In Step 4, we first show that  $Q_{B, \frac{\epsilon}{B(1+\epsilon)}} \subseteq I_{\text{out}}$ .

We assume that  $\frac{1}{1+\epsilon} \|r_j\|_2^2 \leq \|r_j\|_2^2 \leq \|r_j\|_2^2$  always holds; by Proposition 2.15, this happens with high probability. Thus, if  $|c_{i_j}|^2 \geq \frac{\epsilon}{B(1+\epsilon)} \|r_j\|_2^2$ , then  $|c_{i_j}|^2 \geq \frac{\epsilon}{B(1+\epsilon)} \|r_j\|_2^2 \geq \frac{\epsilon}{B(1+\epsilon)} \|r_i\|_2^2$ .

By construction,  $Q_{B, \frac{\epsilon}{B(1+\epsilon)}} \subseteq I$ . A straightforward induction shows that, if  $j \in Q_{B, \frac{\epsilon}{B(1+\epsilon)}}$ , then iteration  $j$  outputs  $t_{i_j}$  and the previous iterations output exactly the set of the  $j$  larger terms in  $I$ .

By Proposition 2.5, since  $I_{\text{out}}$  is a superset of  $Q_{B, \frac{\epsilon}{B(1+\epsilon)}}$ , if  $\tilde{c} = \sum_{j \in I_{\text{out}}} c_{i_j} \delta_{i_j}$ , then  $\|\tilde{c} - c\|_2^2 \leq (1 + \epsilon) \|c_{\text{opt}} - c\|_2^2$ , as desired. ■

Before giving the complete privacy argument, we give a lemma, similar to the above. Suppose a set  $P$  of indices is a subset of another set  $Q$  of indices. We will say that  $P$  is a *prefix* of  $Q$  if  $i \in P, t_j > t_i$ , and  $j \in Q$  imply  $j \in P$ .

**Lemma 3.3** *The output set  $I_{\text{out}}$  is a prefix of  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$  except with probability  $2^{-k}$ .*

**Proof:** Note that  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$  is a subset of  $I$  and  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$  is a prefix of the universe, so  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$  is a prefix of  $I$ . The set  $I_{\text{out}}$  is also a prefix of  $I$ . It follows that, of the sets  $I_{\text{out}}$  and  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$ , one is a prefix of the other (or they are equal).

So suppose, toward a contradiction, that  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$  is a proper prefix of  $I_{\text{out}}$ . Let  $q = |Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}|$ , so  $q$  is the least number such that  $i_q$  is *not* in  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$ . If the protocol halts before considering  $q$ , then  $I_{\text{out}} \subseteq Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$ , a contradiction. So, in particular, we may assume that  $q < B$  (so the for-loop doesn't terminate). Then, by definition of  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$ , we have  $|c_{i_q}|^2 < \frac{\epsilon}{B(1+\epsilon)^2} \sum_{j \geq q} |c_{i_j}|^2$ . It follows that

$$\begin{aligned} |c_{i_q}|^2 &< \frac{\epsilon}{B(1+\epsilon)^2} \sum_{i \geq q} |c_i|^2 \\ &= \frac{\epsilon}{B(1+\epsilon)^2} \|r_q\|_2^2 \\ &\leq \frac{\epsilon}{B(1+\epsilon)} \|r_q\|_2^2. \end{aligned}$$

Thus the protocol halts without outputting  $t_q$ , after outputting exactly the elements in  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$ .  $\blacksquare$

Finally, we turn to privacy.

**Theorem 3.4** *Protocol PRIVATE\_EUCLIDEAN\_HEAVY\_HITTERS leaks no more than  $\|c\|_2^2$  and  $c_{\text{opt}}$ .*

**Proof:** With the random inputs  $R_1$  and  $R_2$  encoded into the output, it is straightforward to show that Protocol PRIVATE\_EUCLIDEAN\_HEAVY\_HITTERS is a private protocol in the traditional sense that the protocol messages leak no more than the inputs and outputs. This is done by composing simulators for PRIVATE-SAMPLE-SUM and SMC. It remains only to show only that we can simulate the joint distribution on  $(\tilde{c}, R_1, R_2)$  given as simulator-input  $c_{\text{opt}}$  and  $\|c\|$ . We will show that  $R_1$  is indistinguishable from independent of the joint distribution of  $(\tilde{c}, R_2)$ , which we will simulate directly.

First, we show that  $R_1$  is independent. Except with probability  $2^{-\Omega(k)}$ , the intermediate set  $I$  is a superset of  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$  and the norm estimation is correct. In that case, the protocol outputs a prefix of  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$  and we get identical output if  $I$  is replaced by  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$ . Also,  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$  can be constructed from  $c_{\text{opt}}$  and  $\|c\|_2$ . Since the protocol proceeds without further reference to  $R_1$ , we have shown that the pair  $(\tilde{c}, R_2)$  is indistinguishable from being independent of  $R_1$ . It remains only to simulate  $(\tilde{c}, R_2)$ .

Note that the output  $\tilde{c}$  does depend non-negligibly on  $R_2$ . If  $|c_{i_j}|^2$  is very close to  $\theta \|r_j\|_2^2$ , then the test  $|c_{i_j}|^2 < \theta \|r_j\|_2^2$  in the protocol may succeed with probability non-negligibly far from 0 and from 1, depending on  $R_2$ , since the distortion guarantee on  $\|r_j\|_2^2$  is only the factor  $(1 \pm \epsilon)$ .

The simulator is as follows. Assume that the terms in  $c_{\text{opt}}$  are  $t_0, t_1, \dots, t_{B-1}$  with decreasing order,  $t_0 > t_1 > \dots > t_{B-1}$ . For each  $j \leq B$ , compute  $E_j = \|c - (t_0 + t_1 + \dots + t_{j-1})\|_2^2 = \|c\|_2^2 - \|t_0 + t_1 + \dots + t_{j-1}\|_2^2$  and then run the NORM\_ESTIMATION simulator on input  $E_j$  and  $\epsilon$  to get a sample from the joint distribution  $(\tilde{E}_j, \bar{R}_2)$ , where  $\tilde{E}_j$  is a good estimate to  $E_j$ . Our

simulator then outputs  $t_{i_j}$  if  $|c_{i_j}|^2 \geq \frac{\epsilon}{B(1+\epsilon)} \tilde{E}_j$ , and halts, otherwise, following the final for-loop of the protocol. Call the output of the simulator  $\tilde{s} = \sum_j t_{i_j} \delta_{i_j}$ .

Again using the fact that a prefix of  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$  is output, if  $j \in Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$ , then  $i_j = j$ ; i.e., the  $j$ 'th largest output term is the  $j$ 'th largest overall, so that, if  $j$  is output, we have  $E_j = \|r_j\|_2^2$ . Thus  $(\tilde{E}_j, \bar{R}_2)$  is distributed indistinguishably from  $(\|r_j\|_2^2, R_2)$ . The protocol finishes deterministically using  $I$  and  $\|r_j\|_2^2$  and the simulator finishes deterministically using  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$  and  $\tilde{E}_j$ , but, since the protocol output is identical if  $I$  is replaced by  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$ , the distributions on output  $(\tilde{c}, R_2)$  of the protocol and  $(\tilde{s}, \bar{R}_2)$  of the simulator are indistinguishable. ■

In summary,

**Theorem 3.5** *Suppose Alice and Bob hold integer-valued vectors  $a$  and  $b$  in  $[-M, M]^N$ , respectively. Let  $B, k$  and  $\epsilon$  be user-defined parameters. Let  $c = a + b$ . Let  $T_{\text{opt}}$  be the set of the largest  $B$  terms in  $c$ . There is an protocol, taking  $a, b, B, k$  and  $\epsilon$  as input, given  $T_{\text{opt}}$  and  $\|c\|_2^2$ , computes a representation  $\tilde{c}$  of at most  $B$  terms such that:*

- $\|\tilde{c} - c\|_2 \leq (1 + \epsilon)\|c_{\text{opt}} - c\|_2$ .
- The algorithm uses  $\text{poly}(N, \log(M), B, k, 1/\epsilon)$  time,  $\text{poly}(\log(N), \log(M), B, k, 1/\epsilon)$  communication, and  $O(1)$  rounds.
- The protocol succeeds with probability  $1 - 2^{-k}$  and leaks only  $c_{\text{opt}}$  and  $\|c\|_2$  with security parameter  $k$ .

**Corollary 3.6** *With the same hypotheses and resource bounds, there is a protocol that computes  $\tilde{c}$  and an approximation  $\|\tilde{c} - c\|_\sim$  to  $\|\tilde{c} - c\|_2$  such that  $\frac{1}{1+\epsilon}\|\tilde{c} - c\|_2 \leq \|\tilde{c} - c\|_\sim \leq \|\tilde{c} - c\|_2$  and the protocol leaks only  $c_{\text{opt}}$  and  $\|\tilde{c} - c\|_2$ .*

**Proof:** Run the main protocol and output also  $\|\tilde{c} - c\|_\sim$ , which is computed in the course of the main protocol. Note that  $\|\tilde{c} - c\|_2^2 = \|c\|_2^2 - \|\tilde{c}\|_2^2$  and both  $\|c\|_2$  and  $\tilde{c}$  are available to the main simulator (as input and output, respectively), so we can modify the main simulator to compute  $\|\tilde{c} - c\|_2^2$  as well. ■

### 3.2 Extension to Taxicab Heavy Hitters

In this section, we show that our result of Euclidean approximation can be extended to approximate taxicab heavy hitters.

**Lemma 3.7** *Let  $\tilde{c}$  be the output of `PRIVATE_EUCLIDEAN_HEAVY_HITTERS`. If  $\|c - \tilde{c}\|_2 \leq (1 + \epsilon)\|c - c_{\text{opt}}\|_2$ , then  $\|c - \tilde{c}\|_1 \leq (1 + \sqrt{B\epsilon})\|c - c_{\text{opt}}\|_1$ .*

**Proof:** Let  $(i, c_i)$  be the largest term which is not in  $Q_{B, \frac{\epsilon}{B(1+\epsilon)^2}}$ . From Theorem 3.5 we know  $(\sum_{i \leq j < B} c_j^2)^{\frac{1}{2}} < \sqrt{\epsilon}(\sum_{B \leq j < N} c_j^2)^{\frac{1}{2}}$ . Using the fact that  $\frac{1}{\sqrt{|\text{supp}(x)|}}\|x\|_1 \leq \|x\|_2 \leq \|x\|_1$  for any signal  $x$ , we get

$$\frac{1}{\sqrt{B}} \sum_{i \leq j < B} |c_j| \leq \left( \sum_{i \leq j < B} c_j^2 \right)^{\frac{1}{2}} \leq \sqrt{\epsilon} \left( \sum_{B \leq j < N} c_j^2 \right)^{\frac{1}{2}} \leq \sqrt{\epsilon} \sum_{B \leq j < N} |c_j|.$$

Thus we have  $\|c - \tilde{c}\|_1 \leq \sum_{i \leq j < N} |c_j| = \sum_{i \leq j < B} c_j + \sum_{B \leq j < N} |c_j| = (\sqrt{\epsilon B} + 1) \sum_{B \leq j < N} |c_j| = (\sqrt{\epsilon B} + 1) \|c - c_{\text{opt}}\|_1$ .  $\blacksquare$

Theorem 3.8 follows directly:

**Theorem 3.8** *Suppose Alice and Bob hold integer-valued vectors  $a$  and  $b$  in  $[-M, M]^N$ , respectively. Let  $B$ ,  $k$  and  $\epsilon$  be userdefined parameters. Let  $c = a + b$ . Let  $T_{\text{opt}}$  be the set of the largest  $B$  terms in  $c$ . There is an protocol, taking  $a$ ,  $b$ ,  $M, N, B, k$  and  $\epsilon$  as input, and computes a representation  $\tilde{c}$  of at most  $B$  terms such that:*

- $\|\tilde{c} - c\|_1 \leq (1 + \epsilon) \|c_{\text{opt}} - c\|_1$ .
- The algorithm uses  $\text{poly}(N, \log(M), B, k, 1/\epsilon)$  time,  $\text{poly}(\log(N), \log(M), B, k, 1/\epsilon)$  communication, and  $O(1)$  rounds.
- The protocol succeeds with probability  $1 - 2^{-k}$  and leaks only  $c_{\text{opt}}$  and  $\|c\|_2$  with security parameter  $k$ .

### 3.3 Extension to other Orthonormal Bases

In this section, we consider other orthonormal bases, such as the Fourier basis. Alice and Bob hold vectors  $a$  and  $b$  as before, and want the  $B$  largest Fourier terms—frequencies and corresponding coefficient values. The exact problem requires  $\Omega(N)$  communication, so they settle for an approximation, namely, they want a  $B$ -term Fourier representation  $\tilde{c}$  such that  $\|\tilde{c} - c\|_2 \leq (1 + \epsilon) \|c_{\text{opt}} - c\|_2$ , where  $c_{\text{opt}}$  is the best possible  $B$ -term Fourier representation.

We note that a straightforward generalization of our main result solves this problem privately and efficiently. Alice and Bob locally compute the inverse Fourier transform  $F^{-1}a$  and  $F^{-1}b$  of their vectors  $a$  and  $b$ . Because the Fourier transform is linear,  $x = F^{-1}c = F^{-1}a + F^{-1}b$ . Alice and Bob now want to compute an approximation to the ordinary heavy hitters for the vector  $x$ . Suppose the result is  $\tilde{x}$ . Then  $\tilde{x}$  is the compact collection of Fourier terms and  $\tilde{c} = F\tilde{x}$  is the corresponding approximate representation of  $c$ . By the Parseval equality, since the Fourier basis is orthogonal, for any  $y$ , we have  $\|y\|_2 = \|Fy\|_2 = \|F^{-1}y\|_2$ . It follows that  $\|\tilde{c} - c\|_2 \leq (1 + \epsilon) \|c_{\text{opt}} - c\|_2$  if and only if  $\|\tilde{x} - x\|_2 \leq (1 + \epsilon) \|x_{\text{opt}} - x\|_2$ , so the algorithm is correct when transformed to the Fourier domain. It also follows that leaking  $\|c\|_2$  is equivalent to leaking  $\|Fc\|_2$ , so the algorithm is private when transformed to the Fourier domain. Alice and Bob require the additional overhead of computing a Fourier transform locally, which fits within the overall budget.

## 4 Lower Bounds

In this Section, we show some lower bounds for problems related to our main problem, such as computing an approximation to  $c_{\text{opt}}$  without leaking  $\|c\|_2$ . The results are straightforward, but we include them to motivate the approximation and leakage of the protocols we present.

**Theorem 4.1** *There is an infinite family of settings of parameters  $M, N, B, k, \epsilon$  such that any protocol that computes the Euclidean norm exactly on the sum  $c$  of individually-held inputs  $a$  and  $b$ , uses communication  $\Omega(N)$ . Similarly, any protocol that computes the exact Heavy Hitters or computes the qualified set  $Q_{c,1,1}$  exactly uses communication  $\Omega(N)$ .*

**Proof:** Consider the set disjointness problem, which requires  $\Omega(N)$  communication [16]. Alice and Bob hold  $\{0, 1\}$ -valued vectors  $a$  and  $b$  of length  $N$  such that each of  $a$  and  $b$  has exactly  $(N/4)$  1's and the supports are either disjoint or intersect in exactly one index. The task is to determine the intersection size. Then, if  $c = a + b$ , we have  $\|c\|_2^2 = N/2$  or  $\|c\|_2^2 = N/2 + 3$ , depending on the size of the intersection, so a protocol for  $\|c\|_2$  can be used to solve the set disjointness problem. Similarly, finding the one largest heavy hitter solves the set disjointness problem.

Now consider vectors of length  $N + 1$  in which indices 0 to  $N - 1$  directly code an instance of set disjointness as above and index  $N$  has a value that is always  $\sqrt{N/2 + 2}$ . Then  $|Q_{c,1,1}| = 1$  or  $|Q_{c,1,1}| = 0$  depending on the norm of indices 0 to  $N - 1$ , which requires communication  $\Omega(N)$  to determine. ■

The above theorem motivates our study of *approximate* heavy hitters, for which there are protocols with exponentially better communication cost than the exact heavy hitters problem. The next theorem motivates leaking the Euclidean norm, by showing that *any* efficient protocol for the approximate heavy hitters problem leaks the Euclidean norm on all instances within a class.

**Theorem 4.2** *There is an infinite family of settings of parameters  $M, N, B, k, \epsilon$  such that any protocol that solves the Euclidean Heavy Hitters problem on the sum  $c$  of individually-held inputs  $a$  and  $b$ , leaking only  $c_{\text{opt}}$ , uses communication  $\Omega(N)$ . Furthermore, for an infinite class of inputs in which  $\|c\|_2$  is not constant, any such protocol either computes  $\|c\|_2$  or uses communication  $\Omega(N)$ .*

**Proof:** Consider vectors  $c$  of one of two cases, given by random permutations of the following vectors:

$$\left\{ \begin{array}{ll} (2N, \overbrace{1, 1, \dots, 1}^{N/2-1}, 0, 0, \dots, 0), & \text{(case 1)} \\ (2N, \overbrace{N, N, \dots, N}^{N/2-1}, 0, 0, \dots, 0), & \text{(case 2).} \end{array} \right.$$

Fix  $B = 1$  and  $\epsilon \gg 1/N$ . A correct protocol finds the top term in case 1. In case 2, it turns out that the correctness requirement is vacuous, but, fortunately, the privacy requirement is useful. A protocol leaking only  $c_{\text{opt}}$  must behave indistinguishably in cases 1 and 2 since  $c_{\text{opt}}$  is the same, so a private protocol reliably finds the top coefficient in case 2. Since a protocol for case 2 can be used to solve the set disjointness problem, such a protocol uses  $\Omega(N)$  bits of communication. In particular, any protocol either behaves differently on the two cases—thereby computing  $\|c\|_2$  for inputs in the union of the two cases—or uses communication  $\Omega(N)$ . ■

Note that the above theorem also shows that it is impossible in some cases to solve the approximate *taxicab* heavy hitters problem efficiently without leaking the *Euclidean* norm.

Although the class of inputs above is contrived, the (implied) parameter settings are natural, *i.e.*,  $\log(M), \log(N), B, k, 1/\epsilon$  can be made to be polynomially related, etc.

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